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# ESTIMATES OF SOME INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. In this article, by a different method without using known identities, some estimates have been obtained for existing inequalities such as midpoint, trapezoid, Jensen, and Simpson, which are important for convex functions whose modulus of the derivatives are convex. These inequalities cover the previously puplished results.

## 1. Introduction

The usefulness of inequalities involving convex functions is realized from the very beginning and is now widely acknowledged as one of the prime driving forces behind the development of several modern branches of mathematics and has been given considerable attention. Some famous results for such estimations consist of Hermite-Hadamard, trapezoid, midpoint, Simpson or Jensen inequalities, ect.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval $I$ of real numbers and $a, b \in I$, with $a < b$. The following double inequality is well known in the literature as the Hermite-Hadamard inequality [8]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities. It gives an estimate from both sides of the mean value of a convex function and also ensure the integrability of convex function. It is also a matter of great interest and one has to note that some of the classical inequalities for means can be obtained from Hadamard's inequality under the utility of peculiar convex functions $f$: These inequalities for convex functions play a crucial role in analysis and as well as in other areas of pure and applied mathematics. The absolute value of the difference of the second part of the (1.1) inequalities is known as the trapezoidal inequality in the literature and was given by Dragomir and Agarwal in 1998 [2]. Then, in 2004, the absolute value of the difference of the first part of the (1.1) inequalities, known as the midpoint inequality by Kirmanci, was given [7]. Thus, these two important inequalities have

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attracted the attention of many readers to date, and many studies have been carried out for different types of convex functions. For recent results and generalizations concerning Hermite-Hadamard's inequalities see [5], [6], [9], [10], [12]-[15], [25] and the references given therein.

In [2], Dragomir and Agarwal proved the following results connected with the right part of (1.1).

**Lemma 1.1.** *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt. \tag{1.2}$$

**Theorem 1.2.** *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{8}\left(|f'(a)|+|f'(b)|\right). \tag{1.3}$$

In [7], Kirmaci proved the following results connected with the left part of (1.1). In [7] some inequalities of Hermite-Hadamard type for differentiable convex mappings were proved using the following lemma.

**Lemma 1.3.** *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on $I^\circ$, $a, b \in I^\circ$ ($I^\circ$ is the interior of $I$) with $a < b$. If $f' \in L([a, b])$, then we have*

$$\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \tag{1.4}$$

$$= (b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (t-1)f'(ta+(1-t)b)dt\right].$$

**Theorem 1.4.** *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)}{8}\left(|f'(a)|+|f'(b)|\right). \tag{1.5}$$

In [1], Dragomir et. al. proved the following some recent developments on Simpson's inequality for which the remainder is expressed in terms of lower derivatives than the fourth.

**Theorem 1.5.** *Suppose $f : [a; b] \to \mathbb{R}$ is an absolutely continuous mapping on $[a, b]$ whose derivative belongs to $L_p[a, b]$. Then, the following inequality holds,*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$\leq \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\left\|f'\right\|_p$$

*where $\frac{1}{p}+\frac{1}{q}=1$.*

In [19], Sarikaya et. al. obtained inequalities for differentiable convex mappings which are connected with Simpson's inequality, and they used the following lemma to prove it.

**Lemma 1.6.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on $I^{\circ}$ such that $f' \in L_1[a,b]$, where $a,b \in I^{\circ}$ with $a<b$, then the following equality holds:*

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx \tag{1.6}$$

$$= \frac{b-a}{2}\int_0^1\left[\left(\frac{t}{2}-\frac{1}{3}\right)f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)+\left(\frac{1}{3}-\frac{t}{2}\right)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.$$

The main inequality in [19], pointed out for $s=1$, as follows:

**Theorem 1.7.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}$ such that $f' \in L_1[a,b]$, where $a,b \in I^{\circ}$ with $a<b$. If $|f'|^q$ is a convex on $[a,b]$, $q>1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{3|f'(b)|^q+|f'(a)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(b)|^q+3|f'(a)|^q}{4}\right)^{\frac{1}{q}}\right\},$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

There are many new studies in the literature on the Simpson type inequalities for different types of convex functions. For the last two decades, extensions, generalizations and refinements have been made for such inequalities, references can be found on these issues [1], [16]-[24].

**Theorem 1.8** (Jensen Inequality)**.** [11] *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $\varphi$ is a real-valued integrable function on $[a,b]$ with $a<b$. Then, we have*

$$f\left(\frac{1}{b-a}\int_a^b \varphi(t)dt\right) \leq \frac{1}{b-a}\int_a^b f(\varphi(t))\,dt.$$

Further, some refinements of Jensen's inequality was proved by Dragomir et. al in [3], [4].

Motivated by the results mentioned above, the purpose of this paper is obtained some inequalities such as midpoint, trapezoid, Jensen, and Simpson, which are important for convex functions in general by a different method without using known identities, and these inequalities cover the previously puplished results.

## 2. Midpoint inequalities

In this section, using the properties of the convexity and by Hölder inequality, we give midpoint inequalities. We begin the following theorem:

**Theorem 2.1.** *Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}$, $a,b \in I^{\circ}$ with $a<b$ and $f' \in L[a,b]$. Then, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \tag{2.1}$$

$$\leq \begin{cases} \dfrac{(b-a)}{8}\left(|f'(a)|+|f'(b)|\right), & \text{if } |f'| \text{ is convex on } [a,b] \\ \dfrac{(b-a)}{2^{2+\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left[\left(\dfrac{|f'(b)|^q+3|f'(a)|^q}{2}\right)^{\frac{1}{q}}\right. & \text{if } |f'|^q \text{ is convex on } [a,b],\ \frac{1}{p}+\frac{1}{q}=1,\ q>1. \\ \left.+\left(\dfrac{|f'(a)|^q+3|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right], & \end{cases}$$

*Proof.* We can write

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\ =& \left|\frac{1}{b-a}\int_a^b \left[f(x)-f\left(\frac{a+b}{2}\right)\right]dx\right| \\ =& \left|\frac{1}{b-a}\int_a^b \int_{\frac{a+b}{2}}^x f'(t)dtdx\right| \\ \leq& \frac{1}{b-a}\int_a^{\frac{a+b}{2}}\int_x^{\frac{a+b}{2}}|f'(t)|\,dtdx + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b\int_{\frac{a+b}{2}}^x |f'(t)|\,dtdx. \end{aligned}$$

By the change of integration order, we get

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\ \leq& \frac{1}{b-a}\int_a^{\frac{a+b}{2}}\int_a^t |f'(t)|\,dxdt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b\int_t^b |f'(t)|\,dxdt \\ =& \frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)|f'(t)|\,dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (b-t)|f'(t)|\,dt. \end{aligned} \tag{2.2}$$

Using the convexity of $|f'|$, we find that

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\ \leq& \frac{2}{(b-a)^2}\int_a^{\frac{a+b}{2}}(t-a)\left[(t-a)\left|f'\left(\frac{a+b}{2}\right)\right| + \left(\frac{a+b}{2}-t\right)|f'(a)|\right]dt \\ &+\frac{2}{(b-a)^2}\int_{\frac{a+b}{2}}^b (b-t)\left[\left(t-\frac{a+b}{2}\right)|f'(b)| + (b-t)\left|f'\left(\frac{a+b}{2}\right)\right|\right]dt \\ =& \frac{2}{(b-a)^2}\left\{\frac{2(b-a)^3}{24}\left|f'\left(\frac{a+b}{2}\right)\right| + \frac{(b-a)^3}{48}|f'(a)| + \frac{(b-a)^3}{48}|f'(b)|\right\} \\ =& \frac{(b-a)}{8}\left(|f'(a)|+|f'(b)|\right) \end{aligned}$$

and the first inequality is proved.

To prove the other half of the inequality in (2.1), using (2.2) and from Hölder inequality we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{1}{b-a}\left(\int_a^{\frac{a+b}{2}} (t-a)^p\,dt\right)^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}} |f'(t)|^q\,dt\right)^{\frac{1}{q}} \\
&+\frac{1}{b-a}\left(\int_{\frac{a+b}{2}}^b (b-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^b |f'(t)|^q\,dt\right)^{\frac{1}{q}} \\
=\ & \frac{(b-a)^{\frac{1}{p}}}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left[\left(\int_a^{\frac{a+b}{2}} |f'(t)|^q\,dt\right)^{\frac{1}{q}}+\left(\int_{\frac{a+b}{2}}^b |f'(t)|^q\,dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Using the convexity of $|f'|^q$, we find that

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{(b-a)^{\frac{1}{p}}}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left\{\left(\frac{2}{b-a}\int_a^{\frac{a+b}{2}}\left[(t-a)\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left(\frac{a+b}{2}-t\right)|f'(a)|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{2}{b-a}\int_{\frac{a+b}{2}}^b\left[\left(t-\frac{a+b}{2}\right)|f'(b)|^q+(b-t)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
=\ & \frac{(b-a)}{2^{2+\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{(b-a)}{2^{2+\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left[\left(\frac{|f'(b)|^q+3|f'(a)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

and the second inequality is proved. □

## 3. Trapezoid inequalities

In this section, we give trapeziod inequalities as follows:

**Theorem 3.1.** *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, $a, b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$. Then, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \tag{3.1}$$

$$\leq \begin{cases} \dfrac{(b-a)}{4}\left(\dfrac{|f'(a)|+|f'(b)|}{2}\right) & \text{if } |f'| \text{ is convex on } [a,b] \\[2ex] \dfrac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left(\dfrac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}} & \text{if } |f'|^q \text{ is convex on } [a,b],\ \frac{1}{p}+\frac{1}{q}=1,\ q>1. \end{cases}$$

*Proof.* We can write

$$
\begin{aligned}
&\left|\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
=&\left|\frac{1}{b-a}\int_a^b\left[\frac{f\left(a\right)+f\left(b\right)}{2}-f(x)\right]dx\right| \\
=&\left|\frac{1}{b-a}\int_a^b\frac{f\left(a\right)-f(x)}{2}dx+\frac{1}{b-a}\int_a^b\frac{f\left(b\right)-f(x)}{2}dx\right| \\
=&\left|\frac{1}{b-a}\int_a^b\int_x^b\frac{f'(t)}{2}dtdx-\frac{1}{b-a}\int_a^b\int_a^x\frac{f'(t)}{2}dtdx\right|.
\end{aligned}
$$

By the change of integration order, we get

$$
\begin{aligned}
&\left|\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (3.2)\\
=&\frac{1}{2\left(b-a\right)}\left|\int_a^b\int_t^b f'(t)dxdt-\int_a^b\int_a^t f'(t)dxdt\right| \\
\leq&\frac{1}{\left(b-a\right)}\int_a^b\left|\frac{a+b}{2}-t\right|\left|f'(t)\right|dt.
\end{aligned}
$$

Using the convexity of $|f'|$, we find that

$$
\begin{aligned}
&\left|\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq&\frac{1}{\left(b-a\right)^2}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)\left[\left(t-a\right)\left|f'(b)\right|+\left(b-t\right)\left|f'\left(a\right)\right|\right]dt \\
&+\frac{1}{\left(b-a\right)^2}\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)\left[\left(t-a\right)\left|f'(b)\right|+\left(b-t\right)\left|f'\left(a\right)\right|\right]dt \\
=&\frac{\left(b-a\right)}{8}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)
\end{aligned}
$$

and the first inequality is proved.

To prove the other half of the inequality in (3.1), using (3.2) and from Hölder inequality we have

$$
\begin{aligned}
&\left|\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq&\frac{1}{\left(b-a\right)}\left(\int_a^b\left|f'(t)\right|^q dt\right)^{\frac{1}{q}} \\
&\times\left(\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)^p dt+\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)^p dt\right)^{\frac{1}{p}}
\end{aligned}
$$

$$= \frac{(b-a)^{\frac{1}{p}}}{2(p+1)^{\frac{1}{p}}} \left( \int_a^b |f'(t)|^q \, dt \right)^{\frac{1}{q}}.$$

Using the convexity of $|f'|^q$, we find that

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right|$$

$$\leq \frac{(b-a)^{\frac{1}{p}}}{2(p+1)^{\frac{1}{p}}} \left( \frac{1}{(b-a)} \int_a^b \left[ (t-a) |f'(b)|^q + (b-t) |f'(a)|^q \right] dt \right)^{\frac{1}{q}}$$

$$= \frac{(b-a)}{2(p+1)^{\frac{1}{p}}} \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}$$

and the second inequality is proved. which this completes the proof of the (3.1). □

## 4. Simpson inequalities

In this section, we give Simpson inequalities for convex functions as follows:

**Theorem 4.1.** *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, $a, b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$. Then, we have*

$$\left| \frac{1}{3} \left[ \frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \tag{4.1}$$

$$\leq \begin{cases} \dfrac{(b-a)}{8} \left( |f'(a)| + |f'(b)| \right), & \text{if } |f'| \text{ is convex on } [a,b] \\ \dfrac{(b-a)}{2^{2+\frac{1}{q}} (p+1)^{\frac{1}{p}}} \left[ \left( \dfrac{|f'(b)|^q + 3|f'(a)|^q}{2} \right)^{\frac{1}{q}} \right. & \\ \left. + \left( \dfrac{|f'(a)|^q + 3|f'(b)|^q}{2} \right)^{\frac{1}{q}} \right], & \text{if } |f'|^q \text{ is convex on } [a,b],\ \frac{1}{p}+\frac{1}{q}=1,\ q>1. \end{cases}$$

*Proof.* We can write

$$\left| \frac{1}{3} \left[ \frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right|$$

$$= \left| \frac{1}{b-a} \int_a^b \left[ \frac{f(a)+f(b)}{6} + \frac{2}{3} f\left(\frac{a+b}{2}\right) - f(x) \right] dx \right|$$

$$= \left| \frac{1}{b-a} \int_a^b \frac{f(a)-f(x)}{6} dx + \frac{1}{b-a} \int_a^b \frac{f(b)-f(x)}{6} dx + \frac{2}{3(b-a)} \int_a^b \left( f\left(\frac{a+b}{2}\right) - f(x) \right) dx \right|$$

$$= \left| \frac{1}{6(b-a)} \int_a^b \int_x^b f'(t)dtdx - \frac{1}{6(b-a)} \int_a^b \int_a^x f'(t)dtdx \right.$$

$$\left. + \frac{2}{3(b-a)} \int_a^{\frac{a+b}{2}} \int_x^{\frac{a+b}{2}} f'(t)dtdx + \frac{2}{3(b-a)} \int_{\frac{a+b}{2}}^b \int_x^{\frac{a+b}{2}} f'(t)dtdx \right|.$$

By the change of integration order, we get

$$\begin{aligned}
&\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (4.2)\\
\leq &\left|\frac{1}{6(b-a)}\int_a^b\int_t^b f'(t)dxdt-\frac{1}{6(b-a)}\int_a^b\int_a^t f'(t)dxdt\right.\\
&\left.+\frac{2}{3(b-a)}\int_a^{\frac{a+b}{2}}\int_a^t f'(t)dtdx+\frac{2}{3(b-a)}\int_{\frac{a+b}{2}}^b\int_b^t f'(t)dtdx\right|\\
\leq &\frac{1}{3(b-a)}\int_a^b\left|\frac{a+b}{2}-t\right|\left|f'(t)\right|dt\\
&+\frac{2}{3(b-a)}\int_a^{\frac{a+b}{2}}(t-a)\left|f'(t)\right|dt+\frac{2}{3(b-a)}\int_{\frac{a+b}{2}}^b(b-t)\left|f'(t)\right|dt\\
\leq &\frac{1}{3(b-a)}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)\left|f'(t)\right|dt\\
&+\frac{1}{3(b-a)}\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)\left|f'(t)\right|dt\\
&+\frac{2}{3(b-a)}\int_a^{\frac{a+b}{2}}(t-a)\left|f'(t)\right|dt+\frac{2}{3(b-a)}\int_{\frac{a+b}{2}}^b(b-t)\left|f'(t)\right|dt.
\end{aligned}$$

Using the convexity of $|f'|$, we find that

$$\begin{aligned}
&\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
\leq &\frac{2}{3(b-a)^2}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)\left[(t-a)\left|f'\left(\frac{a+b}{2}\right)\right|+\left(\frac{a+b}{2}-t\right)\left|f'(a)\right|\right]dt\\
&+\frac{2}{3(b-a)^2}\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)\left[\left(t-\frac{a+b}{2}\right)\left|f'(b)\right|+(b-t)\left|f'\left(\frac{a+b}{2}\right)\right|\right]dt\\
&+\frac{4}{3(b-a)^2}\int_a^{\frac{a+b}{2}}(t-a)\left[(t-a)\left|f'\left(\frac{a+b}{2}\right)\right|+\left(\frac{a+b}{2}-t\right)\left|f'(a)\right|\right]dt\\
&+\frac{4}{3(b-a)^2}\int_{\frac{a+b}{2}}^b(b-t)\left[\left(t-\frac{a+b}{2}\right)\left|f'(b)\right|+(b-t)\left|f'\left(\frac{a+b}{2}\right)\right|\right]dt\\
= &\frac{2}{3(b-a)}\left\{\frac{2(b-a)^3}{48}\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{(b-a)^3}{24}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\right\}\\
&+\frac{4}{3(b-a)}\left\{\frac{2(b-a)^3}{24}\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{(b-a)^3}{48}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\right\}\\
\leq &\frac{(b-a)}{12}\left[\frac{\left|f'(b)\right|+\left|f'(a)\right|}{2}\right]+\frac{(b-a)}{12}\left[\left|f'(b)\right|+\left|f'(a)\right|\right]\\
= &\frac{(b-a)}{8}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)
\end{aligned}$$

and the first inequality is proved.

To prove the other half of the inequality in (4.1), using (4.2) and from Hölder inequality we have

$$
\begin{aligned}
&\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{1}{3(b-a)}\left(\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)^p dt\right)^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}|f'(t)|^q dt\right)^{\frac{1}{q}} \\
&+\frac{1}{3(b-a)}\left(\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^b|f'(t)|^q dt\right)^{\frac{1}{q}} \\
&+\frac{2}{3(b-a)}\left(\int_a^{\frac{a+b}{2}}(t-a)^p dt\right)^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}|f'(t)|^q dt\right)^{\frac{1}{q}} \\
&+\frac{2}{3(b-a)}\left(\int_{\frac{a+b}{2}}^b(b-t)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^b|f'(t)|^q dt\right)^{\frac{1}{q}} \\
=\ & \frac{(b-a)^{\frac{1}{p}}}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left[\left(\int_a^{\frac{a+b}{2}}|f'(t)|^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{a+b}{2}}^b|f'(t)|^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Using the convexity of $|f'|^q$, we find that

$$
\begin{aligned}
&\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)^{\frac{1}{p}}}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left\{\left(\frac{2}{b-a}\int_a^{\frac{a+b}{2}}\left[(t-a)\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left(\frac{a+b}{2}-t\right)|f'(a)|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{2}{b-a}\int_{\frac{a+b}{2}}^b\left[\left(t-\frac{a+b}{2}\right)|f'(b)|^q+(b-t)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
=\ & \frac{(b-a)}{2^{2+\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{(b-a)}{2^{2+\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left[\left(\frac{|f'(b)|^q+3|f'(a)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

and the second inequality is proved. which this completes the proof of the (4.1). □

## 5. Jensen inequalities

**Theorem 5.1.** *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $\varphi$ is a real-valued integrable function on $[a,b]$ with $a<b$. If $f'$ and $\varphi'$ are bounded functions on $[a,b]$, then the following inequality holds:*

$$
\left|f\left(\frac{1}{b-a}\int_a^b\varphi(x)\,dx\right)-\frac{1}{(b-a)}\int_a^b f(\varphi(x))\,dx\right|\leq\frac{\|f'\|_\infty\|\varphi'\|_\infty}{3}(b-a). \quad (5.1)
$$

*Proof.* Let's choose as $u=\frac{1}{b-a}\int\limits_a^b \varphi(x)\,dx$. Using the properties bounded of $f'$ and $\varphi'$, we find that

$$\begin{aligned}
&\left| f\left(\frac{1}{b-a}\int\limits_a^b \varphi(x)\,dx\right) - \frac{1}{(b-a)}\int\limits_a^b f(\varphi(x))\,dx \right| \\
&= \left| \frac{1}{(b-a)}\int\limits_a^b \left[f(u)-f(\varphi(x))\right]dx \right| \\
&= \left| \frac{1}{(b-a)}\int\limits_a^b \int\limits_{\varphi(x)}^{u} f'(t)\,dtdx \right| \\
&\leq \frac{\|f'\|_\infty}{(b-a)}\int\limits_a^b |u-\varphi(x)|\,dx \\
&= \frac{\|f'\|_\infty}{(b-a)}\int\limits_a^b \left| \frac{1}{(b-a)}\int\limits_a^b \left[\varphi(s)-\varphi(x)\right]ds \right| dx \\
&\leq \frac{\|f'\|_\infty}{(b-a)^2}\int\limits_a^b \int\limits_a^b \left| \int\limits_x^s \varphi'(z)\,dz \right| dsdx \\
&\leq \frac{\|f'\|_\infty \|\varphi'\|_\infty}{(b-a)^2}\int\limits_a^b \int\limits_a^b |s-x|\,dsdx.
\end{aligned}$$

By calculated the last above integral, we get

$$\begin{aligned}
&\left| f\left(\frac{1}{b-a}\int\limits_a^b \varphi(x)\,dx\right) - \frac{1}{(b-a)}\int\limits_a^b f(\varphi(x))\,dx \right| \\
&\leq \frac{\|f'\|_\infty \|\varphi'\|_\infty}{(b-a)^2}\int\limits_a^b \left[\int\limits_a^x (x-s)\,ds + \int\limits_x^b (s-x)\,ds\right] dx \\
&= \frac{\|f'\|_\infty \|\varphi'\|_\infty}{2(b-a)^2}\int\limits_a^b \left[(x-a)^2+(b-x)^2\right] dx \\
&= \frac{\|f'\|_\infty \|\varphi'\|_\infty}{3}(b-a)
\end{aligned}$$

which this completes the proof of the (5.1). □

**Theorem 5.2.** *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $\varphi$ is a real-valued integrable function on $[a,b]$ with $a<b$. If $f'$ is bounded function on $[a,b]$, and $|\varphi'|$*

*is convex function on* $[a,b]$*, then the following inequality holds:*

$$\left|f\left(\frac{1}{b-a}\int_a^b \varphi(x)\,dx\right)-\frac{1}{(b-a)}\int_a^b f(\varphi(x))\,dx\right| \leq \frac{\|f'\|_\infty}{3}\left[\frac{|\varphi'(a)|+|\varphi'(b)|}{2}\right](b-a). \tag{5.2}$$

*Proof.* Let's choose as $u=\frac{1}{b-a}\int_a^b \varphi(x)\,dx$. Using the properties bounded of $f'$, we find that

$$\begin{aligned}
&\left|f\left(\frac{1}{b-a}\int_a^b \varphi(x)\,dx\right)-\frac{1}{(b-a)}\int_a^b f(\varphi(x))\,dx\right| \\
=&\left|\frac{1}{(b-a)}\int_a^b [f(u)-f(\varphi(x))]\,dx\right| \\
=&\left|\frac{1}{(b-a)}\int_a^b \int_{\varphi(x)}^u f'(t)\,dtdx\right| \\
\leq&\frac{\|f'\|_\infty}{(b-a)}\int_a^b |u-\varphi(x)|\,dx \\
=&\frac{\|f'\|_\infty}{(b-a)}\int_a^b \left|\frac{1}{(b-a)}\int_a^b [\varphi(s)-\varphi(x)]\,ds\right| dx \\
=&\frac{\|f'\|_\infty}{(b-a)^2}\int_a^b \left|\int_a^b \int_x^s \varphi'(z)\,dzds\right| dx \\
\leq&\frac{\|f'\|_\infty}{(b-a)^2}\int_a^b \left[\int_a^x \int_s^x |\varphi'(z)|\,dzds+\int_x^b \int_x^s |\varphi'(z)|\,dzds\right] dx.
\end{aligned}$$

By the change of integration order, we get

$$\begin{aligned}
&\left|f\left(\frac{1}{b-a}\int_a^b \varphi(x)\,dx\right)-\frac{1}{(b-a)}\int_a^b f(\varphi(x))\,dx\right| \\
\leq&\frac{\|f'\|_\infty}{(b-a)^2}\int_a^b \left[\int_a^x (z-a)|\varphi'(z)|\,dz+\int_x^b (b-z)|\varphi'(z)|\,dz\right] dx \\
=&\frac{\|f'\|_\infty}{(b-a)^2}\left[\int_a^b \int_a^x (z-a)|\varphi'(z)|\,dzdx+\int_a^b \int_x^b (b-z)|\varphi'(z)|\,dzdx\right]
\end{aligned}$$

$$= \frac{2\left\|f'\right\|_{\infty}}{(b-a)^{2}} \int_{a}^{b} (b-z)(z-a)\left|\varphi'(z)\right| dz.$$

Using the convexity of $|\varphi'|$, we find that

$$\left| f\left( \frac{1}{b-a} \int_{a}^{b} \varphi(x)\, dx \right) - \frac{1}{(b-a)} \int_{a}^{b} f(\varphi(x))\, dx \right|$$

$$\leq \frac{2\left\|f'\right\|_{\infty}}{(b-a)^{3}} \int_{a}^{b} (b-z)(z-a)\left[(b-z)\left|\varphi'(a)\right| + (z-a)\left|\varphi'(b)\right|\right] dz$$

$$= \frac{\left\|f'\right\|_{\infty}}{3} \left[ \frac{\left|\varphi'(a)\right| + \left|\varphi'(b)\right|}{2} \right] (b-a)$$

which this completes the proof of the (5.2). □

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